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Limitation of application of the center manifold reduction in aeroelasticity

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Abstract

In this paper, the method of center manifold reduction is applied to the limit cycle calculations of a three-dimensional thin airfoil placed in an incompressible flow. Limit cycle oscillations are caused by a cubic structural restoring force corresponding to the aileron rotation. The equation of motion is written as an integro-differential equation and also as an approximate set of ordinary differential equations. Two different implementations of the method of center manifold reduction for these two cases are briefly outlined. It is emphasized, that the formal power series expansions used in the method of center manifold reduction typically diverge and cause the method not to give satisfactory results. An example is presented, when the method of center manifold reduction cannot even qualitatively predict the occurrence of a stable limit cycle of a small amplitude.

1. Introduction

The method of center manifold reduction was developed in recent decades as a tool for investigation of bifurcations in nonlinear dynamical systems. Such systems arise during modelling a physical system in the form of equations of motion. The most important bifurcation takes place when the steady solution of these equations loses its stability (for example, as a result of a change of the velocity) and a new solution appears (e.g. limit cycle oscillations). The well-known example of such a phenomenon is an aircraft flutter instability known as the Hopf bifurcation (Hassard et al., 1981). The center manifold itself is a certain low-dimensional subspace of the phase space, smooth enough to be called a manifold, containing all asymptotic solutions (trajectories). The center manifold is so important because it has attracting properties, which means that all trajectories with initial conditions lying sufficiently close to this manifold tend to it asymptotically with time. The center manifold is also invariant; in what follows all trajectories with initial conditions placed in this manifold remain in it all the time. Therefore, if only asymptotic solutions of equations of motion are of interest, it is convenient to consider only the low-dimensional center manifold instead of the entire phase space. The procedure of obtaining a low-dimensional system of equations of motion on the center manifold from initial multi-dimensional system is called center manifold reduction. In the case of the Hopf bifurcation, the corresponding center manifold is two-dimensional and the limit cycle oscillations are described by only two ordinary differential equations.

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Nomenclature

a_1, a_2, e_1, e_2 Jones' function parameters (Eq. (18))
b semi-chord (Fig. 1)
F_h, F_α, F_β spring restoring force and moments per unit length
h plunge displacement
C_h, C_α, C_β spring stiffness constants per unit length
c_β nonlinear spring coefficient (Eq. (1))
I_α, I_β moments of inertia per unit length of wing-aileron (relative to x_a) and aileron (relative to x_c), respectively
M_a mass of wing-aileron per unit length
M_w mass of wing-aileron-support per unit length
$r_\alpha = \sqrt{I_\alpha/M_a b^2}$ nondimensional radius of gyration
$r_\beta = \sqrt{I_\beta/M_a b^2}$ nondimensional radius of gyration of aileron
S_α, S_β static moments per unit length of wing-aileron (relative to x_a) and aileron (relative to x_c), respectively
U flow velocity
U_0 linear flutter velocity (bifurcation point)
$u = U - U_0$ bifurcation parameter
x_a nondimensional location of elastic axis (Fig. 1)
x_c nondimensional location of aileron hinge line (Fig. 1)
$x_\alpha = S_\alpha/M_a b$ nondimensional distance from the elastic axis to the center of gravity of the wing-aileron
$x_\beta = S_\beta/M_a b$ nondimensional distance from the hinge line to the center of gravity of the aileron
α pitch angle (Fig. 1)
β aileron rotation angle (Fig. 1)
ρ air density
$\hat{\rho} = \rho b^4$
$\omega_h = \sqrt{C_h/M_w}$ uncoupled natural frequency of plunging motion
$\omega_\alpha = \sqrt{C_\alpha/I_\alpha}$ uncoupled natural frequency of pitching motion
$\omega_\beta = \sqrt{C_\beta/I_\beta}$ uncoupled natural frequency of torsional vibration of the aileron around hinge line
$\zeta_h, \zeta_\alpha, \zeta_\beta$ damping coefficients corresponding to physical degrees of freedom
$\mathbf{u}, \mathbf{q}$ vectors of physical and modal coordinates, respectively

Essentially, the method of center manifold reduction consists of the nonlinear transformations of coordinates which in turn are given in the form of formal power series expansions. At present, this is the only known way of solving the problem effectively, and also the source of certain limitations of application of the center manifold in practice. This is because such series expansions typically diverge and can be used only up to some finite order (Chow and Hale, 1982). The aim of this work is to show examples of aeroelastic systems for which the method of center manifold reduction does not give satisfactory results. These are two- and three-dimensional thin airfoils with special values of parameters. It seems that until new implementation of this method is developed, probably not using formal series expansions, the method of center manifold reduction cannot be an effective and general tool for determining limit cycle oscillations in aeroelastic systems. Nevertheless, the method itself is very attractive: it does not require any simplifying assumptions and gives a low-dimensional system of equations off motion directly for asymptotic solutions, without numerical integration in time (which is practically impossible to carry out in the case of multi-dimensional space of initial conditions).

The center manifold entered subsonic aeroelasticity in the last decade as the method of handling only structural nonlinearities, so far [see Grzędziński (1993a,b, 1994, 1995, 1997, 1999), also Dessi et al. (1999), Liu et al. (1999), Lewis (1995), and others dealing with approximate systems of ordinary differential equations]. A more detailed review of other methods used recently for investigation of nonlinear flutter equation of an airfoil not only with structural nonlinearities is given by Lee et al. (1999). The greatest expectation from center manifold theory was the possibility to calculate limit cycle oscillations without simplifying the unsteady aerodynamics and also without assuming harmonic motion, required in the commonly used harmonic balance analysis, for instance. It is worth noting here that the subsonic nonlinear flutter equation describing the limit cycle oscillations is always of the form of an integro-differential equation, due to the time-history of the unsteady aerodynamic forces. The integro-differential aerodynamic operator itself is linear, but causes some difficulties in nonlinear stability analysis—especially in the time domain—due to the infinite dimension of the corresponding phase space (Hale, 1977). A more general nonlinear form of such an operator including the transonic regime is not known yet (although works on this subject are in progress for many years), so the present work relates only to those aeroelastic systems with structural nonlinearities and linear aerodynamic forces.

In order to evaluate limit cycle oscillation given by the center manifold reduction, any other exact method is necessary for comparison. Unfortunately, there is neither analytical nor general numerical method for solving integro-differential equations, and every numerical method is problem dependent. Moreover, often some additional simplifying assumptions are needed to ensure the computation time is not excessive, as pointed out by Lee and Leblanc (1986). Therefore, since the development of a new numerical method is not the goal of this work, an indirect method has been used for comparison, based on the approximate flutter equation. There exist many approximate models of aeroelastic systems [e.g. Edwards et al. (1979)] which involve rational approximations of aerodynamic forces in the frequency domain and give the equations of motion in the time domain in the form of ordinary differential equations. The possibility of replacing an integro-differential equation by a system of ordinary differential equations depends on the form of a corresponding integral kernel function. Exact requirements for such replacement are given, for example, by Hassard et al. (1981). For ordinary differential equations, there are many numerical tools available, one of them being well suited for calculation of limit cycle oscillations—the continuation and bifurcation software AUTO97 developed by Doedel et al. (1998). The results of the amplitude, frequency and the time history of limit cycle oscillations given by this method applied to the approximate flutter equation will be compared with those of the center manifold reduction.

The application of center manifold theory to an aeroelastic system of N degrees of freedom depends on whether the corresponding equation of motion is of the form of an integro-differential equation or of the form of a system of ordinary differential equations. For the integro-differential equation, the procedure of center manifold reduction contains the following steps:

(i) formulation of the problem in terms of a system of $2N$ nonlinear integro-differential equations of the first order instead of a system of N equations of the second order—this is requirement of methods of bifurcation theory worked out for such equations;
(ii) identification of bifurcation point (i.e. linear flutter velocity $U = U_0$)—this is done by solving the fully linearized flutter equation;
(iii) increasing the number of generalized coordinates by one by adding the difference of velocity $u = U - U_0$ as a new variable and also increasing the number of equations to $2N + 1$ by introducing a new equation $\mathrm{d}u/\mathrm{d}t = 0$—this is done in order to work on interval in velocity space in the vicinity of bifurcation point (otherwise the center manifold exists only for one value of the velocity, $U = U_0$, and vanishes if $U \neq U_0$);
(iv) restriction of the aeroelastic system to the appropriate center manifold—this step requires creation of a special nonlinear transformation of the initial $2N + 1$ dimensional system of integro-differential equations into a two-dimensional system of ordinary differential equations of the first kind;
(v) normalization of the reduced system—this step puts the reduced aeroelastic system into a simpler form by applying so called near-identity change of coordinates; the simplicity achieved lies in a phase-shift symmetry of resulting system of equations;
(vi) calculation of limit cycle amplitude and frequency for a given velocity—this task, because of the symmetry of the final equations, is equivalent to finding roots of a polynomial with real coefficients.

The last three steps deal with formal power series expansions of nonlinear terms with respect to generalized coordinates and, therefore, restrict the analysis to a certain neighborhood of bifurcation point. The area of validity of results and the proper number of terms of the series have to be estimated numerically for each aeroelastic system separately.

For the system of ordinary differential equations the procedure is simpler, because steps 4 and 5 can be performed simultaneously by using a single nonlinear transformation of coordinates.

Before the results are presented, a brief description of the center manifold reduction is given, concerning both integro-differential and ordinary differential equations. A detailed overview of problems related to the above steps of the algorithm is given by Crawford (1991).

2. Three-dimensional airfoil

The nonlinear aeroelastic system under consideration consists of a three-dimensional thin airfoil with aileron, placed in a two-dimensional incompressible flow. The geometry of the airfoil is shown in Fig. 1. Such airfoil was already investigated theoretically and experimentally by Conner et al. (1997), revealing several different types of limit cycle oscillations caused by aileron freeplay. Unfortunately, freeplay cannot be properly treated by the center manifold reduction since all nonlinear functions allowed by this method must be of the form of power series. This is the first limitation of application of the center manifold reduction in aeroelasticity—very restrictive since aileron freeplay is the

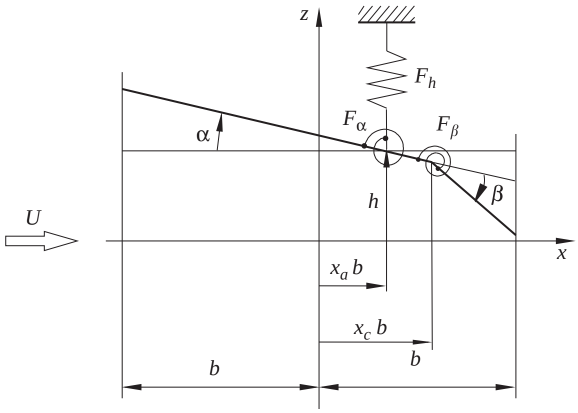

Fig. 1. Thin airfoil with aileron.

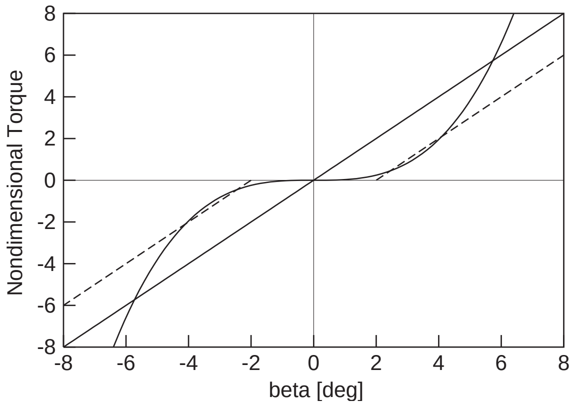

Fig. 2. Restoring moment of aileron rotation.

most common nonlinear phenomenon in aeroelastic systems (due to wear of the material during normal exploitation). In the present work, instead of the aileron freeplay the cubic restoring moment F_β has been assumed

$$F_\beta = C_\beta \beta + c_\beta \beta^3, \tag{1}$$

where $C_\beta = 0$, and c_β is a constant coefficient. The plot of this function compared to a four-degree freeplay is shown in Fig. 2. Note that the linear term in Eq. (1) equals zero, in what follows the linearized system has one rigid degree of freedom (aileron rotation). The remaining springs in the plunge and pitch degree of freedom are assumed linear,

$$F_h = C_h h, \quad F_\alpha = C_\alpha \alpha, \tag{2}$$

where C_h and C_α are known constants.

Displacements of the airfoil during an unsteady motion are described by the three-dimensional vector of physical coordinates $\mathbf{u}(t)$ being function of time t:

$$\mathbf{u}(t) = \begin{Bmatrix} h(t) \\ \alpha(t) \\ \beta(t) \end{Bmatrix}. \tag{3}$$

The flutter equation written in physical coordinates is as follows:

$$\mathbf{M}_u \ddot{\mathbf{u}}(t) + \mathbf{B}_u \dot{\mathbf{u}}(t) + \mathbf{K}_u \mathbf{u}(t) + \mathbf{k}_u(\mathbf{u}) = \mathbf{f}_A^{(u)}, \tag{4}$$

where the mass, damping and stiffness matrices, $\mathbf{M}_u$, $\mathbf{B}_u$ and $\mathbf{K}_u$, respectively, are defined in Appendix A, and $\mathbf{k}_u(\mathbf{u})$ is the nonlinear term generated by Eq. (1).

For an arbitrary motion, the vector of unsteady aerodynamic forces is given by a convolution integral

$$\mathbf{f}_A^{(u)} = \frac{\rho U^2}{2}\int_{-\infty}^{0} \mathbf{G}_u(-\tau)\mathbf{u}\left(t + \frac{b}{U}\tau\right)\mathrm{d}\tau, \tag{5}$$

where the matrix $\mathbf{G}_u(-\tau)$ is composed of response functions corresponding to the impulsive changes of physical coordinates. For a thin airfoil in a two-dimensional incompressible flow, these functions can be expressed in terms of well-known Wagner's function (Fung, 1955).

In the absence of aerodynamic and damping forces and under the assumption that all springs are linear, the natural frequencies ω_j and modes $\boldsymbol{\varphi}_j$ ($j = 1, 2, 3$) can be calculated from the eigenvalue problem

$$\omega_j^2\mathbf{M}_u\boldsymbol{\varphi}_j = \mathbf{K}_u\boldsymbol{\varphi}_j. \tag{6}$$

Note that $\omega_3 = 0$ because the linear term in Eq. (1) equals zero.

The vector $\mathbf{q}(t)$ of modal coordinates is defined by the relation

$$\mathbf{u}(t) = \boldsymbol{\Phi}\mathbf{q}(t), \tag{7}$$

where the square matrix $\boldsymbol{\Phi}$ is composed of eigenvectors of the eigenproblem given by Eq. (6).

The flutter equation written in modal coordinates is as follows:

$$\ddot{\mathbf{q}}(t) + \mathbf{B}_q\dot{\mathbf{q}}(t) + \mathbf{K}_q\mathbf{q}(t) + \mathbf{k}_q(\mathbf{q}) = \mathbf{f}_A^{(q)}, \tag{8}$$

where $\mathbf{f}_A^{(q)}$ is the vector of generalized unsteady aerodynamic forces,

$$\mathbf{f}_A^{(q)} = \frac{\rho U^2}{2}\int_{-\infty}^{0} \mathbf{G}_q(-\tau)\mathbf{q}\left(t + \frac{b}{U}\tau\right)\mathrm{d}\tau, \tag{9}$$

the diagonal matrix $\mathbf{K}_q$ is composed of squares of eigenfrequencies ω_j^2, and the remaining matrices are given by

$$\mathbf{G}_q(-\tau) = \boldsymbol{\Phi}^{\mathrm{T}}\mathbf{G}_u(-\tau)\boldsymbol{\Phi}, \tag{10}$$

$$\mathbf{B}_q = \boldsymbol{\Phi}^{\mathrm{T}}\mathbf{B}_u\boldsymbol{\Phi}. \tag{11}$$

The nonlinear term is written as

$$\mathbf{k}_q(\mathbf{q}) = \boldsymbol{\Phi}^{\mathrm{T}}\left\{\begin{array}{c} 0 \\ 0 \\ c_\beta\left(\varphi_3^{(1)}q_1 + \varphi_3^{(2)}q_2 + \varphi_3^{(3)}q_3\right)^3 \end{array}\right\}, \tag{12}$$

where $\varphi_i^{(j)}$ denotes the ith component of the eigenvector $\boldsymbol{\varphi}_j$. Local bifurcation theory of dynamical systems (Chow and Hale, 1982; Hassard et al., 1981) has been developed for the first-order equations. By introducing a six-dimensional vector of new coordinates $\mathbf{y}(t)$,

$$\mathbf{y}(t) = \left\{\begin{array}{c} \mathbf{q}(t) \\ \dot{\mathbf{q}}(t) \end{array}\right\}, \tag{13}$$

the first-order flutter equation is obtained

$$\dot{\mathbf{y}}(t) = \mathscr{L}_y\mathbf{y}(t) + \mathbf{k}_y(\mathbf{y}), \tag{14}$$

where the linear integral operator $\mathscr{L}_y\mathbf{y}$ is

$$\mathscr{L}_y\mathbf{y}(t) = \mathbf{D}_y\mathbf{y}(t) + \int_{-\infty}^{0}\mathbf{G}_y(-\Theta; U)\mathbf{y}(t+\Theta)\,\mathrm{d}\Theta \tag{15}$$

and square matrices of order 6, $\mathbf{D}_y$, $\mathbf{G}_y$, and the nonlinear term $\mathbf{k}_y(\mathbf{y})$ are given by

$$\mathbf{D}_y = \begin{bmatrix} 0 & \mathbf{I} \\ -\mathbf{K}_y & -\mathbf{B}_y \end{bmatrix},$$

$$\mathbf{G}_y(-\Theta; U) = \begin{bmatrix} 0 & 0 \\ (\rho U^3/2b)\mathbf{G}_q\left(-\frac{U}{b}\Theta\right) & 0 \end{bmatrix},$$

$$\mathbf{k}_y(\mathbf{y}) = \begin{Bmatrix} 0 \\ -\mathbf{k}_q(\mathbf{q}) \end{Bmatrix}. \tag{16}$$

The further analysis consists of the investigation of stability of the steady solution $\mathbf{y} = 0$ of Eq. (14). Two different methods will be used: direct reduction of the integro-differential equation (14) on the center manifold, as described by Grzędziński (1993a), and also investigation of the system of ordinary differential equations, obtained in place of Eq. (14) by using an approximation of the Wagner's function. Both methods give the limit cycle oscillation in the neighborhood of each bifurcation point (flutter velocity). For comparison, the set of approximate ordinary differential equations will be also reduced on the center manifold, in addition to the numerical analysis performed by using the continuation and bifurcation software AUTO97.

3. Approximate set of ordinary differential equations

For a thin airfoil, the response matrix function $\mathbf{G}_q(-\tau)$ can be expressed analytically in terms of the Wagner's function $\phi(\tau)$. This gives the following expression for the unsteady aerodynamic forces (Fung, 1955) under the assumption that all terms arising from initial conditions are damped out, and aerodynamic forces do not depend on time explicitly:

$$\begin{aligned}\mathbf{f}_A^{(q)} = &-\hat{\rho}\Bigg(\hat{\mathbf{M}}_{nc}\ddot{\mathbf{q}} + \frac{U}{b}\mathbf{P}_1\dot{\mathbf{q}} + \left(\frac{U}{b}\right)^2\mathbf{P}_0\mathbf{q} \\ &+ \left(\frac{U}{b}\right)^2\hat{\mathbf{R}}_{s1}\int_0^t \frac{\mathrm{d}\phi((U/b)\tau)}{\mathrm{d}\tau}\mathbf{q}(t-\tau)\,\mathrm{d}\tau + \frac{U}{b}\hat{\mathbf{R}}_{s2}\int_0^t \frac{\mathrm{d}^2\phi((U/b)\tau)}{\mathrm{d}\tau^2}\mathbf{q}(t-\tau)\,\mathrm{d}\tau\Bigg),\end{aligned} \tag{17}$$

where $\hat{\rho} = \rho b^4$. The matrices $\mathbf{P}_0$ and $\mathbf{P}_1$ depend on eigenmodes, geometry of an airfoil, and initial values of the Wagner's function, $\phi(0)$ and $\mathrm{d}\phi(0)/\mathrm{d}\tau$, while remaining matrices depend on eigenmodes and geometry of an airfoil.

It is known (Hassard et al., 1981) that, if the matrix function $\mathbf{G}_q(-\tau)$ in Eq. (9) satisfies a certain ordinary differential equation with constant coefficients, then it is possible to transform Eq. (8) into a system of ordinary differential equations by introducing appropriate new variables. Such a condition is satisfied, if an exponential approximation of the Wagner's function $\phi(\tau)$ is used, given by Jones (1940)

$$\phi(\tau) = 1 - a_1\mathrm{e}^{-e_1\tau} - a_2\mathrm{e}^{-e_2\tau}, \tag{18}$$

where $a_1 = 0.165$, $a_2 = 0.335$, $e_1 = 0.0455$, and $e_2 = 0.3$. After introducing the vector of new variables

$$\mathbf{y}(t) = \begin{Bmatrix} \mathbf{q}_1(t) \\ \mathbf{q}_2(t) \\ \mathbf{w}_1(t) \\ \mathbf{w}_2(t) \end{Bmatrix}, \tag{19}$$

where

$$\mathbf{q}_1(t) = \mathbf{q}(t), \quad \mathbf{q}_2(t) = \dot{\mathbf{q}}(t), \tag{20}$$

$$\mathbf{w}_j(t) = \mathbf{A}_j\int_0^t \mathrm{e}^{-e_j(U/b)\tau}\mathbf{q}(t-\tau)\,\mathrm{d}\tau \quad (j = 1, 2), \tag{21}$$

the flutter equation, Eq. (8), takes the form of a set of 12 ordinary differential equations of the first order,

$$\dot{\mathbf{y}}(t) = \mathbf{C}(U)\mathbf{y}(t) + \mathbf{v}_y(\mathbf{y}), \tag{22}$$

where dependence on the velocity (which acts as a bifurcation parameter) is expressed explicitly as a polynomial of the third degree

$$\mathbf{C}(U) = \mathbf{C}_0 + \mathbf{C}_1\frac{U}{b} + \mathbf{C}_2\left(\frac{U}{b}\right)^2 + \mathbf{C}_3\left(\frac{U}{b}\right)^3 \tag{23}$$

and the structural nonlinearities are described by the vector $\mathbf{v}_y(\mathbf{y})$. All matrices, vectors and parameters appearing in Eqs. (17), (21), and (23) are given in Appendix B.

Limit cycles described by Eq. (22) can be calculated by using the continuation and bifurcation software AUTO97. The AUTO97 software is mainly designated for performing a limited bifurcation analysis of algebraic systems and of systems of ordinary differential equations of the form

$$\dot{\mathbf{y}}(t) = \mathbf{f}(\mathbf{y}(t), U),$$

where $\mathbf{y}$ and $\mathbf{f}$ are real vector functions and U denotes free parameter. For any given steady or periodic solution $\mathbf{y}_0(t)$ corresponding to the fixed value of parameter $U = U_0$ the program can compute the stable and unstable branches of solution $\mathbf{y}(t)$ for $U > U_0$ by using the continuation method. For each step of the parameter U, the Floquet multipliers are computed in order to determine stability along these branches. If a Hopf bifurcation occurs at certain point. the program automatically generates starting data for the computation of periodic orbits (limit cycles). This feature of AUTO97 software was extensively used in the present analysis of aeroelastic systems. In addition to a Hopf bifurcation, the program can detect folds, branch points, period doubling bifurcations, and bifurcation to tori. Each new branch of periodic solution is computed in a separate run of the program. At least two runs are required to compute one limit cycle branch: first the Hopf bifurcation point must by located during continuation of the steady solution $\mathbf{y} = 0$, and then the limit cycle branch can be computed. Since AUTO97 discretizes the boundary value problem for ordinary differential equations by the method of orthogonal collocation using piecewise polynomials in each mesh interval, it solves in each step a system of nonlinear algebraic equations. Moreover, because each step of the parameter U is small, one-point Newton iteration is applied with initial solution taken from the previous step. The program is therefore fast, but computing limit cycles by using the center manifold reduction is much faster.

4. Center manifold reduction—ordinary differential equations

If any bifurcation occurs in a dynamical system, then the phase space splits in general into three manifolds: stable—generated by eigenvalues λ with $\mathrm{Re}(\lambda) < 0$; unstable—generated by eigenvalues with $\mathrm{Re}(\lambda) > 0$; and center manifold, corresponding to $\mathrm{Re}(\lambda) = 0$ (Kelley, 1967). Center manifold is invariant, locally attracting and asymptotically stable. Moreover, it is of finite dimension—for the Hopf bifurcation it is two dimensional. It means that in the space of all solutions of Eq. (14) or Eq. (22), the bifurcating solution tends asymptotically to a two-dimensional attracting subspace. The asymptotic solution (limit cycle oscillations) satisfies a certain system of two nonlinear ordinary differential equations of the first order, which can be derived either from the integro-differential equation, Eq. (14), or from the ordinary differential equation, Eq. (22), written for many degrees of freedom. The main advantage of using the center manifold reduction is just a small number of variables describing an asymptotic motion.

In the simplest case of the ordinary differential equation, Eq. (22), the steady solution $\mathbf{y}(t) = 0$ is stable for $U = 0$ and all eigenvalues λ of the matrix $\mathbf{C}(U)$,

$$\mathbf{C}(U)\hat{\mathbf{y}} = \lambda\hat{\mathbf{y}}, \tag{24}$$

are either real and negative or complex-conjugate with negative real parts. When the parameter U increases, it may reach a certain critical value U_0 at which a complex-conjugate, pure imaginary pair of eigenvalues appears

$$\lambda_{1,2} = \pm \mathrm{i}\omega_0 \tag{25}$$

with corresponding complex-conjugate eigenvectors

$$\hat{\mathbf{y}}_{1,2} = \hat{\mathbf{y}}_{\mathrm{re}} \pm \mathrm{i}\hat{\mathbf{y}}_{\mathrm{im}}, \tag{26}$$

and the steady solution loses its stability and bifurcates asymptotically to the limit cycle oscillations contained in the center manifold which is tangent to the linear subspace spanned by the real and imaginary parts of the eigenvectors $\hat{\mathbf{y}}_{1,2}$. This phenomenon is known as the Hopf bifurcation. The value U_0 is the critical flutter speed of a linear aeroelastic system. However, there is one subtlety here: a locally attracting center manifold is defined at only a single point $U = U_0$, at which the new oscillatory solution has still zero amplitude. For $U > U_0$, the center manifold vanishes since there are no more complex-conjugate imaginary eigenvalues. Consequently, it is impossible to work on an interval in parameter space about $U = U_0$. To solve this problem, the center manifold reduction is applied to the so-called suspended system obtained from Eq. (22) by considering the velocity a variable, not a parameter. Thus the number of variables increases by one additional variable

$$u = U - U_0 \tag{27}$$

and an additional equation for this variable is simply

$$\dot{u} = 0. \tag{28}$$

The suspended system is then

$$\dot{\mathbf{x}}(t) = \mathbf{A}_0\mathbf{x}(t) + \mathbf{v}_x(\mathbf{x}), \tag{29}$$

where

$$\mathbf{x}(t) = \begin{Bmatrix} \mathbf{y}(t) \\ u \end{Bmatrix}, \tag{30}$$

$$\mathbf{A}_0 = \begin{bmatrix} \mathbf{C}(U) & 0 \\ 0 & 0 \end{bmatrix} \tag{31}$$

and the nonlinear vector $\mathbf{v}_x(\mathbf{x})$ is easily obtained according to the foregoing. The center manifold for the suspended system is three dimensional and the velocity u can now vary as a variable without affecting the existence of the center manifold.

The center manifold reduction can be applied to Eq. (29) in a number of ways. The most effective is the method described in details in Chow and Hale (1982), and briefly outlined below. It is always possible to rearrange the sequence of variables and split the vector $\mathbf{x}$ into two vectors $\mathbf{x}_c$ and $\mathbf{x}_s$

$$\mathbf{x} = \begin{Bmatrix} \mathbf{x}_c \\ \mathbf{x}_s \end{Bmatrix}$$

in such way that $x_3 = u$ ($x_3 \equiv x_{c3}$, the last component of the vector $\mathbf{x}_c$) and rewrite Eq. (29) accordingly in the form

$$\begin{aligned} \dot{\mathbf{x}}_c(t) &= \mathbf{A}_c\mathbf{x}_c(t) + \mathbf{v}_c(\mathbf{x}_c, \mathbf{x}_s), \\ \dot{\mathbf{x}}_s(t) &= \mathbf{A}_s\mathbf{x}_s(t) + \mathbf{v}_s(\mathbf{x}_c, \mathbf{x}_s), \end{aligned} \tag{32}$$

where $\mathbf{x}_c$ and $\mathbf{x}_s$ are vectors of dimension n_c, n_s ($n_c + n_s = M$), respectively (for the particular case of Eq. (29), $n_c = 3$, $n_s = 10$, $M = 13$), and the $n_c \times n_c$ matrix $\mathbf{A}_c$ has two imaginary eigenvalues given by Eq. (25) and one zero eigenvalue. In most cases, it is possible to diagonalize the matrix $\mathbf{A}_0$ and consequently also the matrices $\mathbf{A}_c$ and $\mathbf{A}_s$. The real parts of all eigenvalues of the matrix $\mathbf{A}_s$ are negative, so if the system were linear, the asymptotic motion would be described by the equation

$$\dot{\mathbf{x}}_c(t) = \mathbf{A}_c\mathbf{x}_c(t),$$

because $\mathbf{x}_s \to 0$ as $t \to \infty$. The idea of center manifold reduction is to find such transformation of coordinates $\mathbf{x} \to \boldsymbol{\zeta}$ ($\mathbf{x}_c \to \boldsymbol{\zeta}_c$, $\mathbf{x}_s \to \boldsymbol{\zeta}_s$) that retains the above property in the resulting equation for $\boldsymbol{\zeta}_c$, and also makes that equation invariant when $\boldsymbol{\zeta}_s = 0$. Such an equation will then describe the asymptotic motion in nonlinear case. To solve this problem, two additional assumptions are necessary.

The first assumption is a very strong one, namely that there exists a formal power series representation of the nonlinear vector $\mathbf{v}_x(\mathbf{x})$ in Eq. (29)

$$\mathbf{v}_x(\mathbf{x}) = \sum_{\mu \geqslant 2} \frac{1}{\mu!}\mathbf{V}_\mu \mathbf{x}^\mu, \tag{33}$$

where

$$\mathbf{x}^\mu = \left\{x_1^{\mu_1} \cdot x_2^{\mu_2} \cdots x_M^{\mu_M}\right\}, \quad \sum_{j=1}^{M} \mu_j = \mu, \quad \mu_j \geqslant 0. \tag{34}$$

The number of components of the vector $\mathbf{x}^\mu$ and also the number of columns of each matrix $\mathbf{V}_\mu$ varies from one term to another and equals the number $c_{\mu,M}$ of compositions of the number μ into M parts

$$c_{\mu,M} = \begin{pmatrix} \mu + M - 1 \\ \mu - 1 \end{pmatrix}. \tag{35}$$

It follows from Eq. (33) that also the nonlinear terms $\mathbf{v}_c$ and $\mathbf{v}_s$ have formal power series representations of the same form.

The second assumption is that

$$\sum_{j=1}^{n_c} \lambda_j^{(c)}\mu_j - \lambda_k^{(s)} \neq 0 \tag{36}$$

for $k = 1, 2, \ldots, n_s$ and for each composition of μ ($\mu \geqslant 2$), where $\lambda_j^{(c)}$ and $\lambda_k^{(s)}$ are the eigenvalues of the matrices $\mathbf{A}_c$ and $\mathbf{A}_s$, respectively. This assumption is always satisfied (maybe except for certain very special cases, which must be treated separately). Under the above assumptions, the differential equation (32) can be transformed into the following system of equations in new coordinates $\boldsymbol{\zeta}$ ($\boldsymbol{\zeta}_c$, $\boldsymbol{\zeta}_s$):

$$\begin{aligned}\dot{\boldsymbol{\zeta}}_c(t) &= \mathbf{A}_c\boldsymbol{\zeta}_c(t) + \tilde{\mathbf{v}}_c(\boldsymbol{\zeta}_c, \boldsymbol{\zeta}_s),\\ \dot{\boldsymbol{\zeta}}_s(t) &= \mathbf{A}_s\boldsymbol{\zeta}_s(t) + \tilde{\mathbf{v}}_s(\boldsymbol{\zeta}_c, \boldsymbol{\zeta}_s),\end{aligned} \tag{37}$$

with

$$\tilde{\mathbf{v}}_s(\boldsymbol{\zeta}_c, 0) = 0. \tag{38}$$

The equation corresponding to $\boldsymbol{\zeta}_s = 0$:

$$\dot{\boldsymbol{\zeta}}_c(t) = \mathbf{A}_c\boldsymbol{\zeta}_c(t) + \tilde{\mathbf{v}}_c(\boldsymbol{\zeta}_c, 0) \tag{39}$$

is the final equation on the center manifold and describes the asymptotic motion. The transformation of variables $\mathbf{x} \to \boldsymbol{\zeta}$ is a near-identity transformation

$$\mathbf{x}(t) = \boldsymbol{\zeta}(t) + \sum_{\mu \geqslant 2} \frac{1}{\mu!}\mathbf{B}_\mu \boldsymbol{\zeta}_c^\mu(t), \tag{40}$$

such that its nonlinear part depends only on the vector $\boldsymbol{\zeta}_c$ composed of the first n_c coordinates of the vector $\boldsymbol{\zeta}$. The matrices $\mathbf{B}_\mu$ can be calculated numerically in such way (Chow and Hale, 1982) that not only Eq. (38) is satisfied but in addition Eq. (39) has a very special structure called normal form, and because of its rotational symmetry, in polar coordinates r, θ,

$$\zeta_1(t) = r(t)e^{i\theta(t)}, \quad \zeta_2(t) = \bar{\zeta}_1(t), \quad \zeta_3 = u, \tag{41}$$

can be written as a partially uncoupled system of equations

$$\dot{r} = r\left(\gamma(u) + \sum_{j=1}^{\infty} a_j(u) r^{2j}\right), \tag{42}$$

$$\dot{\theta} = \omega(u) + \sum_{j=1}^{\infty} b_j(u) r^{2j}, \tag{43}$$

where $\gamma(u) \pm i\omega(u)$ is the pair of complex-conjugate eigenvalues ($\gamma(0) = 0$, $\omega(0) = \omega_0$). Eq. (42) does not depend on θ, so it can be solved separately. All functions $\gamma(u)$, $\omega(u)$, $a_j(u)$, $b_j(u)$ are real and have the form of power series expansions with respect to u. In Chow and Hale (1982) there is also presented a detailed numerical algorithm for computing the matrices $\mathbf{B}_\mu$ up to any order, so no tedious algebraic operations are necessary. In practical calculations, Eqs. (42) and (43) are implemented up to some finite order $n (j \leqslant n)$. Therefore, the amplitude r_H of the limit cycle oscillations satisfies an algebraic equation obtained from Eq. (42) by setting $\dot{r} = 0$:

$$\gamma(u) + \sum_{j=1}^{n} a_j(u) r_H^{2j} = 0. \tag{44}$$

For any given u, the left-hand side of Eq. (44) is of the form of a polynomial in r_H. Hence, all possible limit cycle amplitudes are determined by the real positive roots of this polynomial. Since limit cycle oscillations $\zeta_1 = \zeta_H(t)$ on the center manifold are purely harmonic (Hassard et al., 1981),

$$\zeta_H(t) = r_H e^{i\omega_H t}, \tag{45}$$

then for each amplitude r_H the corresponding frequency ω_H is calculated from

$$\omega_H = \omega(u) + \sum_{j=1}^{n} b_j(u) r_H^{2j}. \tag{46}$$

The lowest order of variables Eq. (44) can be truncated to is two, which corresponds to $n = 1$ and gives the amplitude of the limit cycle

$$r_H = \sqrt{-\frac{1}{a_1(0)}\left(\frac{\mathrm{d}\gamma(0)}{\mathrm{d}u}u + \frac{1}{2}\frac{\mathrm{d}^2\gamma(0)}{\mathrm{d}u^2}u^2\right)}. \tag{47}$$

Note that the classical Hopf bifurcation theory (Hassard et al., 1981) gives a more simplified formula,

$$r_H = \sqrt{-\frac{1}{a_1(0)}\frac{\mathrm{d}\gamma(0)}{\mathrm{d}u}u}, \tag{48}$$

which does not contain the term proportional to u^2.

5. Center manifold reduction—integro-differential equations

The problem of existence of the center manifold when the velocity varies, described in the previous section, affects also integro-differential equations. Therefore, instead of Eq. (14) the suspended system is used

$$\dot{\mathbf{x}}(t) = \mathscr{L}_x\mathbf{x}(t) + \mathbf{h}(\mathbf{x}), \tag{49}$$

where the vector $\mathbf{x}(t)$ is given by Eq. (30) in which the vector $\mathbf{y}(t)$ is that of Eq. (13). The linear integral operator $\mathscr{L}_x$ corresponds to the linear flutter speed U_0:

$$\mathscr{L}_x\mathbf{x}(t) = \mathbf{D}_x\mathbf{x}(t) + \int_{-\infty}^{0}\mathbf{G}_x(-\Theta; U_0)\mathbf{x}(t+\Theta)\,\mathrm{d}\Theta, \tag{50}$$

where the nonlinear term is given by

$$\mathbf{h}(\mathbf{x}) = \mathbf{k}_x(\mathbf{x}) + \sum_{\mu\geqslant 2}\frac{1}{(\mu-1)!}\int_{-\infty}^{0}\frac{\mathrm{d}^{\mu-1}\mathbf{G}_x(-\Theta; U_0)}{\mathrm{d}U^{\mu-1}}\mathbf{x}^{\mu}(t+\Theta)\,\mathrm{d}\Theta \tag{51}$$

and

$$\mathbf{D}_x = \begin{bmatrix} 0 & \mathbf{I} & 0 \\ -\mathbf{K}_q & -\mathbf{B}_q & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$\mathbf{G}_x(-\Theta; U) = \begin{bmatrix} 0 & 0 & 0 \\ \frac{\rho U^3}{2b}\mathbf{G}_q\left(-\frac{U}{b}\Theta\right) & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$\mathbf{k}_x(\mathbf{x}) = \begin{Bmatrix} 0 \\ \mathbf{k}_q(\mathbf{q}) \\ 0 \end{Bmatrix}. \tag{52}$$

The series in Eq. (51) is the nonlinear part of the Taylor expansion of the matrix-function $\mathbf{G}_x(-\Theta; U)$ with respect to the velocity U in the neighborhood of U_0.

The procedure of center manifold reduction applied to the integro-differential equation, Eq. (49), described up to the first order in Hassard et al. (1981) and also in details up to any order in Grzędziński (1993a), differs significantly from that for ordinary differential equations. The reason is that the operator $\mathscr{L}_x$ maps the space $\mathtt{C}^-$ of continuous functions $\boldsymbol{\varphi}(\Theta)$ defined over the interval $\Theta \in (-\infty, 0]$ onto the Euclidean space, and the eigenvalue problem $\mathscr{L}_x\boldsymbol{\varphi} = \lambda\boldsymbol{\varphi}$ cannot be even posed directly, because each side of this equality belongs to a different space. Moreover, the problem is really of infinite dimensions since the space of initial conditions is an infinite functional space. This is due to the influence of the history of motion expressed by a convolution integral with infinite delay. The method of solving these problems is given in Hale (1977) and Hassard et al. (1981) and is based on the extension of the operator $\mathscr{L}_x$ in order to map a space of continuous functions onto itself. The extended integral operator is

$$\mathscr{L}\boldsymbol{\varphi}(\Theta) = \begin{cases} \dfrac{\mathrm{d}\boldsymbol{\varphi}(\Theta)}{\mathrm{d}\Theta}, & \text{for } -\infty < \Theta < 0, \\ \mathbf{D}_x\boldsymbol{\varphi}(0) + \int_{-\infty}^{0} \mathbf{G}_x(-\tau; U_0)\boldsymbol{\varphi}(\tau)\,\mathrm{d}\tau & \text{for } \Theta = 0. \end{cases} \tag{53}$$

After introducing the following notation:

$$\mathbf{x}_t(\Theta) = \mathbf{x}(t+\Theta), \tag{54}$$

the nonlinear flutter equation takes the form

$$\dot{\mathbf{x}}_t(\Theta) = \mathscr{L}\mathbf{x}_t(\Theta) + \mathscr{R}\mathbf{x}_t(\Theta), \tag{55}$$

where the nonlinear term is

$$\mathscr{R}\mathbf{x}_t(\Theta) = \begin{cases} 0, & \text{for } -\infty < \Theta < 0, \\ \mathbf{h}(\mathbf{x}_t(0)) & \text{for } \Theta = 0. \end{cases} \tag{56}$$

First note that the time t is considered a parameter and Eq. (55) is written for unknown continuous function $\mathbf{x}_t(\Theta)$ of the argument $\Theta \in (-\infty, 0]$. For $\Theta \in (-\infty, 0)$ Eq. (55) gives the obvious relation $\mathrm{d}\mathbf{x}_t(\Theta)/\mathrm{d}t = \mathrm{d}\mathbf{x}_t(\Theta)/\mathrm{d}\Theta$ while for $\Theta = 0$ gives the flutter equation, Eq. (49). Note also that although the function $\mathbf{x}_t(\Theta)$ is continuous over the interval $(-\infty, 0]$, the functions $\mathscr{L}\mathbf{x}_t(\Theta)$ and $\mathscr{R}\mathbf{x}_t(\Theta)$ may have a jump at $\Theta = 0$, which means that actually both operators $\mathscr{L}$ and $\mathscr{R}$ act from the space $\mathtt{C}^-$ into a wider space.

At the bifurcation point ($U = U_0$), the operator $\mathscr{L}$ has a pure imaginary pair of eigenvalues $\pm \mathrm{i}\omega_0$ plus one zero eigenvalue, and the corresponding eigenvectors span a three-dimensional linear subspace $\mathtt{E}_c$ which is tangent to the three-dimensional center manifold which in turn contains an asymptotic motion. The idea of center-manifold reduction consists of splitting the vector $\mathbf{x}_t(\Theta)$ into two parts

$$\mathbf{x}_t(\Theta) = \mathbf{v}(z_1, z_2, z_3, \Theta) + \mathbf{w}(t, \Theta) \tag{57}$$

in such way, that when the vector $\mathbf{x}_t(\Theta)$ remains on the center-manifold, the vector $\mathbf{v}(z_1, z_2, z_3, \Theta)$ belongs all the time to the subspace $\mathtt{E}_c$ and depends only on three new variables $z_1(t)$, $z_2(t)$ and $z_3(t)$ being functions of time ($z_3(t) \equiv u$). It follows from the properties of the center manifold that the vector $\mathbf{w}(t, \Theta)$ must satisfy the following conditions:

$$\mathbf{w}(t, \Theta) = \mathbf{w}(\mathbf{z}(t), \Theta), \quad \mathbf{w}(0, \Theta) = 0, \quad \frac{\mathrm{d}\mathbf{w}(0, \Theta)}{\mathrm{d}\mathbf{z}} = 0, \tag{58}$$

where $\mathbf{z}(t) = [z_1(t)\ \ z_2(t)\ \ z_3(t)]^{\mathrm{T}}$. The above conditions reflect invariant properties of the center manifold.

Both vectors $\mathbf{v}$ and $\mathbf{w}$ in Eq. (57) must be orthogonal in a certain sense, and obtaining the relation between them is the essence of the method. The orthogonality is defined by the so-called outer product (Hassard et al., 1981)

$$\langle \mathbf{x}^*, \mathbf{x} \rangle = \bar{\mathbf{x}}^{*\mathrm{T}}(0)\mathbf{x}(0) - \int_{-\infty}^{0} \int_{0}^{\eta} \bar{\mathbf{x}}^{*\mathrm{T}}(\xi - \eta)\mathbf{G}_x(-\eta; U_0)\mathbf{x}(\xi)\,\mathrm{d}\xi\,\mathrm{d}\eta, \tag{59}$$

with two continuous functions $\mathbf{x}(\xi)$ and $\mathbf{x}^*(\eta)$ defined over intervals $\xi \in (-\infty, 0]$ and $\eta \in [0, +\infty)$, respectively. The adjoint operator $\mathscr{L}^*$ is defined in a standard way by the relation $\langle \mathbf{x}^*, \mathscr{L}\mathbf{x} \rangle = \langle \mathscr{L}^*\mathbf{x}^*, \mathbf{x} \rangle$. The eigenvalues and eigenfunctions of two eigenproblems

$$\mathscr{L}\boldsymbol{\varphi} = \lambda\boldsymbol{\varphi} \tag{60}$$

and

$$\mathscr{L}^*\boldsymbol{\psi} = \lambda^*\boldsymbol{\psi} \tag{61}$$

satisfy the equalities $\lambda^* = \bar{\lambda}$, $\langle \boldsymbol{\psi}_k, \boldsymbol{\varphi}_l \rangle = \delta_{kl}$. By using Eqs. (58) and (59) and assuming the following form of the vector $\mathbf{x}_t(\Theta)$

$$\mathbf{x}_t(\Theta) = \sum_{j=1}^{3} z_j(t)\boldsymbol{\varphi}_j(\Theta) + \mathbf{w}(\mathbf{z}(t), \Theta), \tag{62}$$

the simple set of three nonlinear first-order ordinary differential equations describing asymptotic motion on the center manifold is obtained

$$\dot{\mathbf{z}} = \boldsymbol{\Lambda}\mathbf{z} + \bar{\boldsymbol{\Psi}}^{\mathrm{T}}(0)\mathbf{h}_0(\mathbf{z}, \mathbf{w}(\mathbf{z}, 0)), \tag{63}$$

where $\boldsymbol{\varphi}_j(\Theta)$ are the eigenfunctions of Eq. (60), the matrix $\boldsymbol{\Psi}$ is composed of the corresponding eigenfunctions $\boldsymbol{\psi}_j(\Theta)$ ($\Theta = 0$, $j = 1, 2, 3$) of Eq. (61), and $\boldsymbol{\Lambda}$ denotes the diagonal matrix of eigenvalues $\mathrm{i}\omega_0, -\mathrm{i}\omega_0, 0$. The vector $\mathbf{w}(\mathbf{z}, \Theta)$ satisfies the integro-differential equation

$$\dot{\mathbf{w}} - \mathscr{L}\mathbf{w} = \begin{cases} -\sum_{j=1}^{3} \bar{\boldsymbol{\psi}}_j^{\mathrm{T}}(0)\mathbf{h}_0(\mathbf{z}, \mathbf{w})\boldsymbol{\varphi}_j(\Theta) & \text{for } -\infty < \Theta < 0, \\ -\sum_{j=1}^{3} \bar{\boldsymbol{\psi}}_j^{\mathrm{T}}(0)\mathbf{h}_0(\mathbf{z}, \mathbf{w})\boldsymbol{\varphi}_j(0) + \mathbf{h}_0(\mathbf{z}, \mathbf{w}) & \text{for } \Theta = 0. \end{cases} \tag{64}$$

Eqs. (62) and (63) are coupled by the right-hand size nonlinear term

$$\mathbf{h}_0(\mathbf{z}, \mathbf{w}) = \mathbf{h}\left(\sum_{j=1}^{3} z_j(t)\boldsymbol{\varphi}_j(0) + \mathbf{w}(\mathbf{z}, 0) \right). \tag{65}$$

The problem of obtaining the function $\mathbf{w}(\mathbf{z}, \Theta)$ from Eq. (64) and also determining the right-hand side of Eq. (63) is solved under assumption that there exists a formal power series representation of the nonlinear vector $\mathbf{h}(\mathbf{x})$ in Eq. (49)

$$\mathbf{h}(\mathbf{x}) = \sum_{\mu \geqslant 2} \frac{1}{\mu!} \mathbf{H}_\mu \mathbf{x}^\mu. \tag{66}$$

It can be shown that under this assumption Eq. (63) takes the form

$$\dot{\mathbf{z}} = \boldsymbol{\Lambda}\mathbf{z} + \sum_{\mu \geqslant 2} \frac{1}{\mu!} \mathbf{D}_\mu \mathbf{x}^\mu, \tag{67}$$

where $\boldsymbol{\Lambda}$ denotes, as before, the diagonal matrix of eigenvalues $\mathrm{i}\omega_0, -\mathrm{i}\omega_0,\ 0$, and $\mathbf{D}_\mu$ are rectangular matrices composed of the complex numbers calculated according to the algorithm of center manifold reduction described in Grzędziński (1993a).

At this stage, the whole problem is reduced to that of transforming Eq. (67) to the normal form by using the near-identity transformation given by Eq. (40), and then applying the procedure described at the end of the previous section in order to find the limit cycle amplitude and frequency.

The most important feature of the algorithm of center manifold reduction (Grzędziński, 1993a) is that the impulsive response matrix $\mathbf{G}_x(-\tau; U)$ does not appear in the algorithm explicitly but in the form of the integrals

$$\int_{-\infty}^{0} \frac{\mathrm{d}^l \mathbf{G}_x(-\Theta; U_0)}{\mathrm{d}U^l} \Theta^j \mathrm{e}^{s\Theta}\, \mathrm{d}\Theta = \frac{\partial^{l+j} \mathbf{A}_x(s, U_0)}{\partial U^l \partial s^j}, \tag{68}$$

where $l \geqslant 0$, $j \geqslant 0$, and the only nonzero block of the matrix

$$\mathbf{A}_x(s, U_0) = \begin{bmatrix} 0 & 0 & 0 \\ \mathbf{A}(s, U_0) & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

is the aerodynamic matrix given by the Laplace transform

$$\mathbf{A}(s, U) = \frac{\rho U^2}{2} \int_0^\infty \mathbf{G}_q(\tau) \mathrm{e}^{-(sb/U)\tau}\, \mathrm{d}\tau \tag{69}$$

and calculated for the critical velocity $U = U_0$ and a pure harmonic motion ($s = \pm \mathrm{i}k\omega_0$, $k = 1, 2, \ldots$). For a three-degree of freedom thin airfoil, the aerodynamic matrix is given in Appendix C (Theodorsen, 1935).

Eq. (68) does not only represent the formal equality but also the way of regularization of the improper integrals, which do not exist in a common sense since the impulsive response matrix function decays too slowly with $\Theta \to \infty$. In addition, the algorithm of center manifold reduction is formulated as a pure numerical algorithm and does not require any symbolic operations.

6. Results

The numerical values of system parameters used in calculations are listed in Table 1. They are essentially the same as those of Conner et al. (1997), except for the damping and nonlinear spring model. The eigenfrequencies of the linearized system (with free aileron rotation) equal 0, 4.55, and 9.93 Hz. There are three bifurcation points (flutter velocities) at 6.93, 10.13, 19.36 m/s, with corresponding frequencies of 4.26, 9.19, and 5.20 Hz. The lowest flutter velocity is taken as a reference velocity U_0.

Table 1
Values of parameters

Parameter	Value
b	0.127 m
x_a	−0.5
x_c	0.5
M_a	1.558 kg/m
M_w	3.3843 kg/m
x_α	0.434
r_α^2	0.536
x_β	0.02
r_β^2	0.013
ω_h	28.9378 rad/s
ω_α	52.7975 rad/s
ω_β	7.6758×10^{-5} rad/s
c_β	391.6579 N
I_α	0.01347 kg m
I_β	0.0003264 kg m
ζ_h	0.0008279
ζ_α	0.002588
ζ_β	0.001830
ρ	1.225 kg/m^3

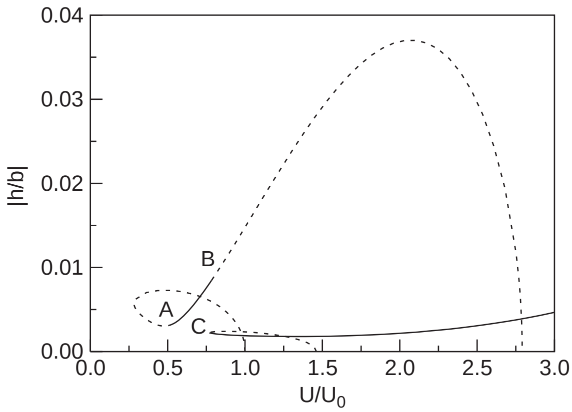

Fig. 3. Limit cycle amplitude in plunge.

First, the limit cycles were calculated by using the AUTO97 software applied directly to the approximate system of ordinary differential equations, Eq. (22), based on Jones' approximation of Wagner's function. The results are shown in Figs. 3–8 (note that only limit cycles are shown). The amplitudes of oscillations in plunge, pitch, and aileron rotation are shown in Figs. 3, 4, and 5, respectively. The values of the amplitude are defined as the highest absolute values achieved during one cycle. Although such an amplitude is not the appropriate measure for solutions not being nearly harmonic in time, as shown in Fig. 7 for time series in pitch, this measure seems to be satisfactory in this particular case since the multiple peak solution lies far beyond any comparison with center manifold solutions. There are two different branches, one of them connecting two different bifurcation points. The stable limit cycles are plotted with solid lines, as that between points A and B, and also that passing to the right from the point C. The unstable limit cycle branches are plotted with broken lines. The corresponding limit cycle frequencies are shown in Fig. 6. The time plots of the two stable branches differ significantly, as shown in Figs. 7 and 8 (where T denotes the period of oscillations). The high frequency limit cycle is nearly harmonic, while the low frequency limit cycle has more complex time history of pitch oscillations.

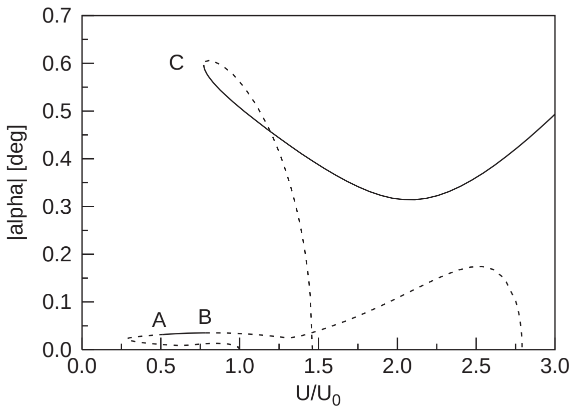

Fig. 4. Limit cycle amplitude in pitch.

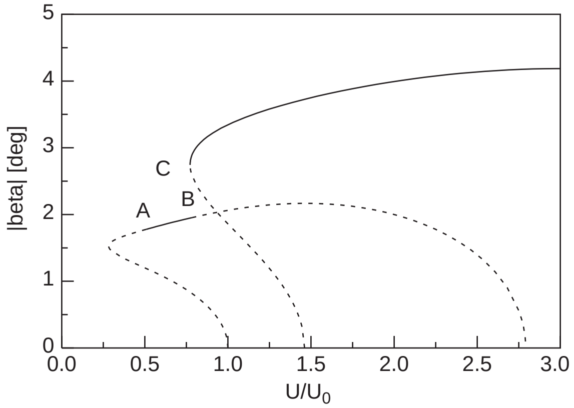

Fig. 5. Limit cycle amplitude in aileron rotation.

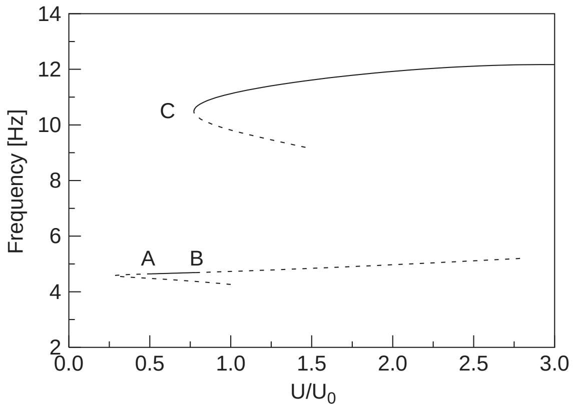

Fig. 6. Limit cycle frequency.

The method of center manifold reduction has been applied three times in order to calculate limit cycles in the neighborhood of the three bifurcation points. All limit cycle solutions calculated this way constitute unstable branches. The sequences of five plots of limit cycle amplitude corresponding to the first five terms ($n = 1, 2, \ldots, 5$) in the series of

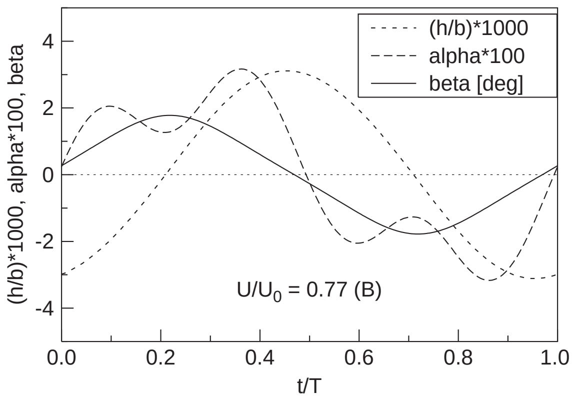

Fig. 7. Time history of low frequency limit cycle.

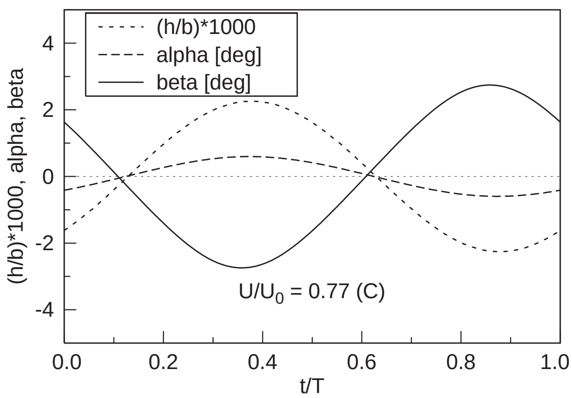

Fig. 8. Time history of high frequency limit cycle.

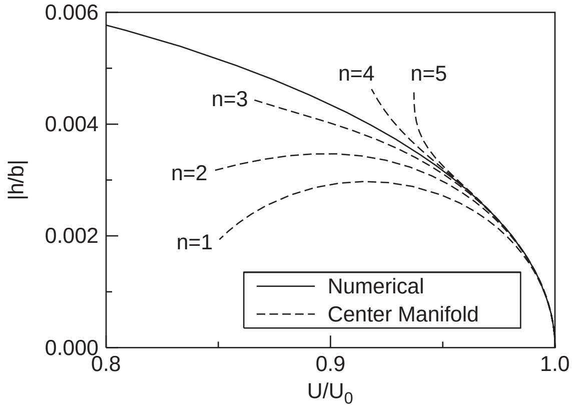

Fig. 9. Sequence of limit cycle amplitudes in plunge.

Eq. (44), and corresponding to the lowest flutter velocity (6.93 m/s) are shown in Figs. 9–11. It can be seen that the interval of velocity over which the results are acceptable is very small, and does not exceed 5% of the linear flutter velocity. This is not enough to predict the stable limit cycle branch. Very similar behavior is observed in the neighborhood of remaining two bifurcation points. The final comparison of limit cycle amplitudes calculated by using the method of center manifold reduction (for $n = 3$, because three-term expansion seems to give the best accuracy in this

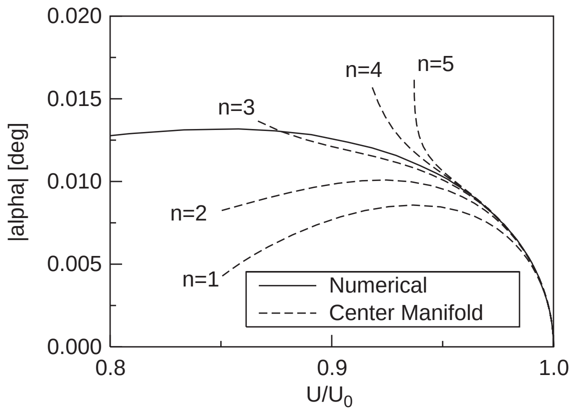

Fig. 10. Sequence of limit cycle amplitudes in pitch.

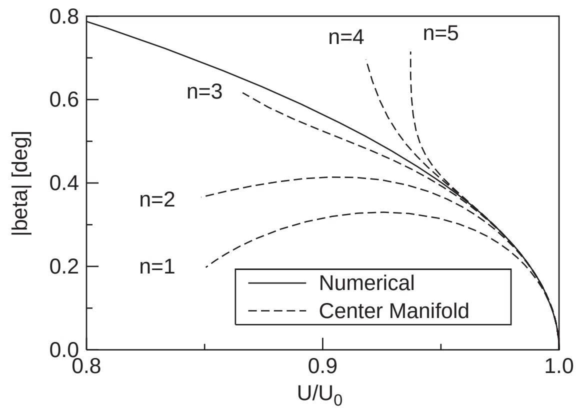

Fig. 11. Sequence of limit cycle amplitudes in aileron rotation.

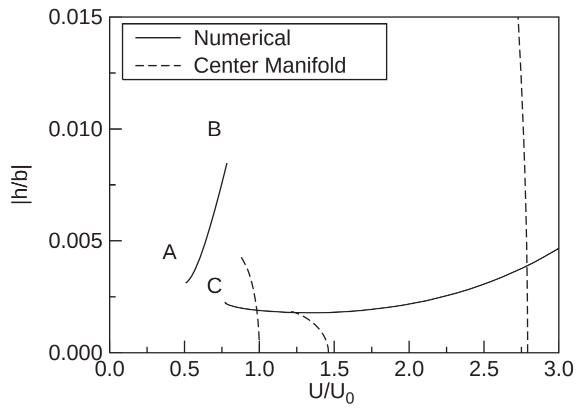

Fig. 12. Limit cycle amplitude in plunge.

particular case, according to Figs. 9–11) with the stable branches given by a pure numerical method (AUTO97) is shown in Figs. 12–14. The conclusion is straightforward—the method of center manifold reduction is not a proper tool for that particular aeroelastic system.

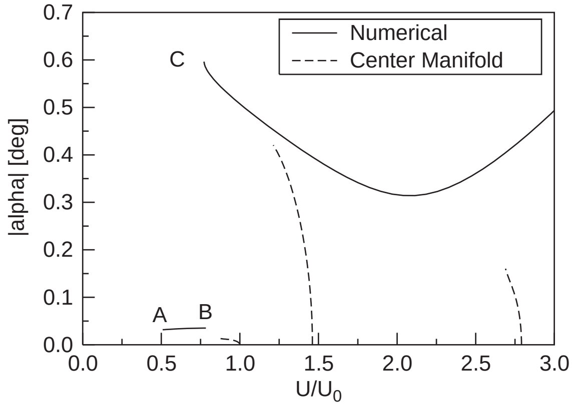

Fig. 13. Limit cycle amplitude in pitch.

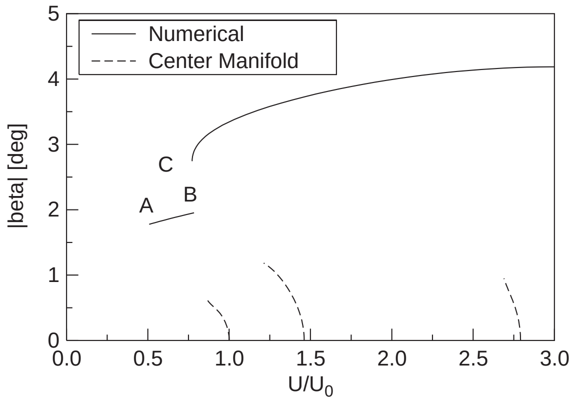

Fig. 14. Limit cycle amplitude in aileron rotation.

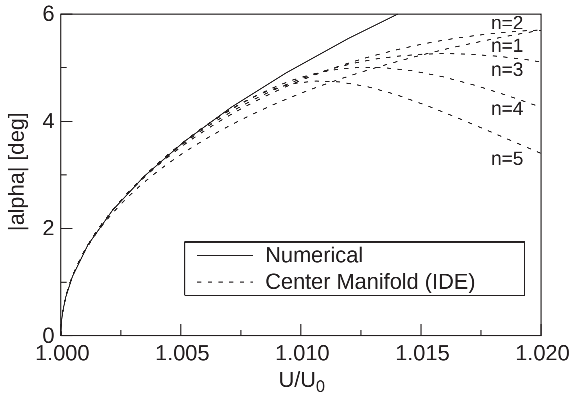

Fig. 15. Sequence of limit cycle amplitudes in pitch for 2-D airfoil.

7. Two-dimensional airfoil: comments on certain results

A two-dimensional airfoil is the simplest and most frequently used nonlinear aeroelastic system when developing new methods for limit cycle calculations.

The preliminary results assumed only a cubic structural nonlinearity in the pitch degree of freedom

$$F_\alpha = C_\alpha(\alpha + c_\alpha \alpha^3).$$

The amplitude of limit cycle oscillations of such system with $c_\alpha = 3$, calculated by the method of center manifold reduction applied to the integro-differential flutter equation (IDE), Eq. (14), is shown in Fig. 15. The remaining system parameters are: $M_w/\pi\rho b^2 = 100$, $x_a = -0.5$, $x_\alpha = 0.25$, $r_\alpha = 0.5$, $\omega_h/\omega_\alpha = 0.2$. Such system was already investigated in Grzędziński (1993b) (center manifold reduction) and Liu et al. (1999) (center manifold reduction combined with perturbation technique). The sequence of five center manifold plots in Fig. 15 correspond to the first five terms ($n = 1, 2, \ldots, 5$) in the series of Eq. (44). Analogous plots calculated by the method of center manifold reduction applied to the approximate set of ordinary differential flutter equations (ODE) based on Jones' approximation of Wagner's function, Eq. (22), are shown in Fig. 16. Numerical results are those of AUTO97 software. The comparison of these three approaches is shown in Fig. 17. Once again, the interval of velocities over which the results of the center manifold reduction are acceptable is very small, and does not exceed 1% of the linear flutter velocity.

It is interesting and somewhat surprising that the simplest estimation of the limit cycle amplitude given by Eq. (48) agrees very good with numerical result over a much wider interval of the velocities. The corresponding plot (denoted as Hopf) is shown in Fig. 18. However, this agreement—deemed a very encouraging result of the center manifold reduction—is misleading, because the amplitude is not the only parameter of the problem, and other parameters do not

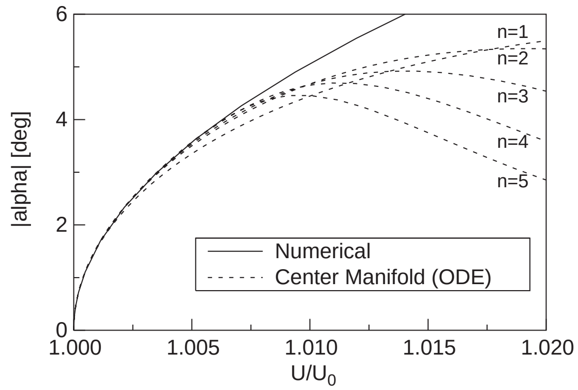

Fig. 16. Sequence of limit cycle amplitudes in pitch for 2-D airfoil.

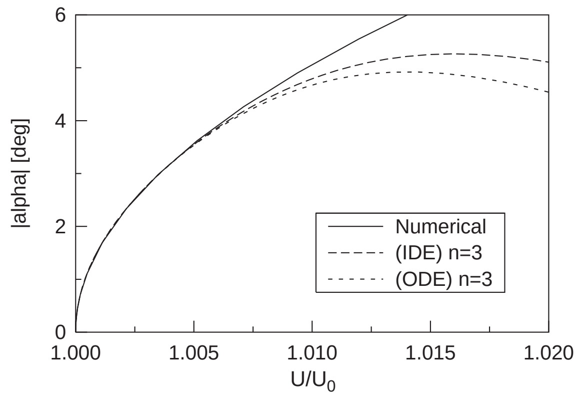

Fig. 17. Limit cycle amplitude—numerical and center manifold reduction.

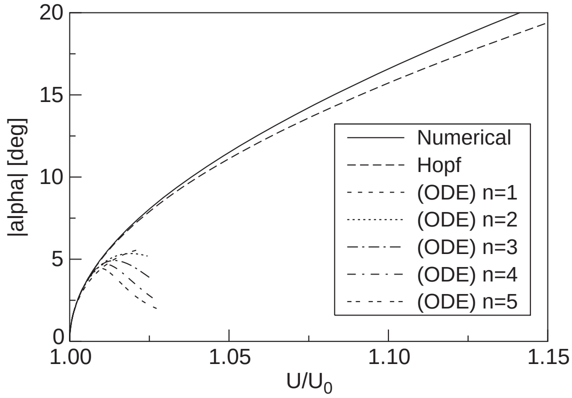

Fig. 18. Limit cycle amplitude in pitch for 2-D airfoil.

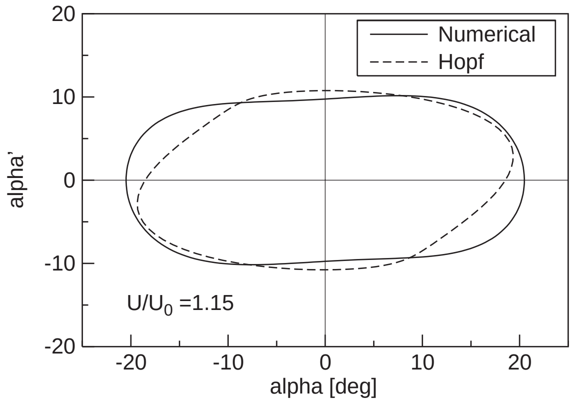

Fig. 19. Phase plot—pitch angle versus angular velocity in pitch.

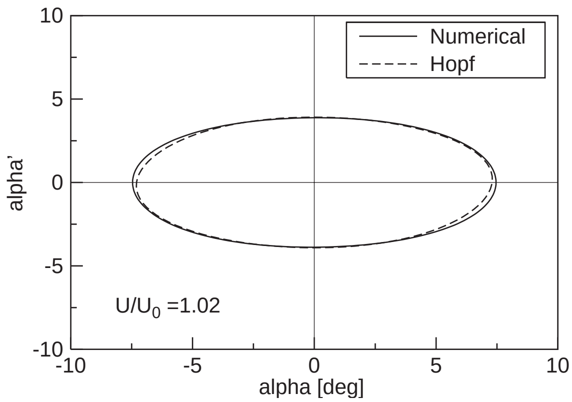

Fig. 20. Phase plot—pitch angle versus angular velocity in pitch.

agree so well. In Fig. 19, the phase space section is shown of angular velocity in pitch versus pitch angle, corresponding to the velocity ratio $U/U_0 = 1.15$. It is clearly visible that the phase plot corresponding to the simplified Hopf formula is nonphysical since the airfoil rotates in the same direction while the angular velocity changes in its sign (at two plot points of intersection with horizontal axis). Similar behavior is still visible (in much smaller scale, however) for the velocity ratio $U/U_0 = 1.02$ (Fig. 20). Consequently, the interval of acceptable agreement once again does not exceed approximately 1% of the linear flutter velocity.

8. Conclusions

Although conceptually the method of center manifold reduction is very suitable for limit cycle calculations, it suffers from the manner of its implementation based on formal power series expansions. These series often diverge and behave like asymptotic series. Moreover, the area of applicability of such series cannot be predicted before performing calculations. Therefore, in order to estimate this area it is necessary to calculate at least the first three terms of the series.

There exist aeroelastic systems (as shown in this paper) for which the method of center manifold reduction does not give even qualitatively acceptable results. In such cases the method of center manifold reduction can serve only as a tool for preliminary local analysis. Generally, the method of center manifold reduction cannot be considered as a reliable method for limit cycle calculations in nonlinear dynamical systems.

Appendix A. Matrices appearing in equations of motion

$$\mathbf{M}_u = M_a b^2 \begin{bmatrix} \dfrac{M_w}{M_a b^2} & -\dfrac{x_\alpha}{b} & -\dfrac{x_\beta}{b} \\ -\dfrac{x_\alpha}{b} & r_\alpha^2 & r_\beta^2 + x_\beta(x_c - x_a) \\ -\dfrac{x_\beta}{b} & r_\beta^2 + x_\beta(x_c - x_a) & r_\beta^2 \end{bmatrix},$$

$$\mathbf{B}_u = \begin{bmatrix} 2M_w\omega_h\zeta_h & 0 & 0 \\ 0 & 2I_\alpha\omega_\alpha\zeta_\alpha & 0 \\ 0 & 0 & 2I_\beta\omega_\beta\zeta_\beta \end{bmatrix},$$

$$\mathbf{K}_u = M_a b^2 \begin{bmatrix} \dfrac{M_w\omega_h^2}{M_a b^2} & 0 & 0 \\ 0 & r_\alpha^2\omega_\alpha^2 & 0 \\ 0 & 0 & r_\beta^2\omega_\beta^2 \end{bmatrix}.$$

Appendix B. Matrices appearing in approximate flutter equation

$$\mathbf{C}_0 = \begin{bmatrix} 0 & \mathbf{I} & 0 & 0 \\ -\mathbf{D}_{nc}^{-1}\mathbf{K}_q & -\mathbf{D}_{nc}^{-1}\mathbf{B}_q & 0 & 0 \\ -\dfrac{1}{\pi}\mathbf{A}_1 & 0 & 0 & 0 \\ -\dfrac{1}{\pi}\mathbf{A}_2 & 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{C}_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \hat{\rho}\mathbf{D}_{nc}^{-1}\mathbf{P}_1 & 0 & 0 \\ 0 & 0 & -e_1\mathbf{I} & 0 \\ 0 & 0 & 0 & -e_2\mathbf{I} \end{bmatrix},$$

$$\mathbf{C}_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ \hat{\rho}\mathbf{D}_{nc}^{-1}\mathbf{P}_0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{C}_3 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -\pi\hat{\rho}\mathbf{D}_{nc}^{-1} & -\pi\hat{\rho}\mathbf{D}_{nc}^{-1} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\mathbf{v}_y(\mathbf{y}) = \begin{Bmatrix} 0 \\ -\mathbf{D}_{nc}^{-1}\mathbf{k}_q(\mathbf{q}) \\ 0 \\ 0 \end{Bmatrix},$$

$$\mathbf{D}_{nc} = \mathbf{I} - \hat{\rho}\hat{\mathbf{M}}_{nc},$$

$$\mathbf{P}_0 = \hat{\mathbf{K}}_{nc} + (1 - a_1 - a_2)\hat{\mathbf{R}}_{s1} + (a_1 e_1 + a_2 e_2)\hat{\mathbf{R}}_{s2},$$

$$\mathbf{P}_1 = \hat{\mathbf{B}}_{nc} + (1 - a_1 - a_2)\hat{\mathbf{R}}_{s2},$$

$$\mathbf{A}_j = a_j e_j \hat{\mathbf{R}}_{s1} - a_j e_j^2 \hat{\mathbf{R}}_{s2} \quad (\text{for } j = 1, 2)$$

$$\mathbf{M}_{nc} = \begin{bmatrix} -\pi & \pi x_a & T_1 \\ \pi x_a & -\pi\left(x_a^2 + \dfrac{1}{8}\right) & -2T_{13} \\ T_1 & -2T_{13} & \dfrac{T_3}{\pi} \end{bmatrix},$$

$$\mathbf{B}_{nc} = \begin{bmatrix} 0 & -\pi & T_4 \\ 0 & \pi\left(x_a - \dfrac{1}{2}\right) & -T_{16} \\ 0 & -T_{17} & -\dfrac{T_{19}}{\pi} \end{bmatrix},$$

$$\mathbf{K}_{nc} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -T_{15} \\ 0 & 0 & -\dfrac{T_{18}}{\pi} \end{bmatrix},$$

$$\mathbf{R}_{s1} = \mathbf{R}_1\mathbf{S}_1, \quad \mathbf{R}_{s2} = \mathbf{R}_1\mathbf{S}_2,$$

$$\mathbf{R}_1 = \begin{bmatrix} -2\pi & 2\pi\left(x_a + \dfrac{1}{2}\right) & -T_{12} \end{bmatrix}^{\mathrm{T}}$$

$$\mathbf{S}_1 = \begin{bmatrix} 0 & 1 & \dfrac{T_{10}}{\pi} \end{bmatrix},$$

$$\mathbf{S}_2 = \begin{bmatrix} 1 & \dfrac{1}{2} - x_a & \dfrac{T_{11}}{2\pi} \end{bmatrix};$$

the general matrix notation: $\hat{\mathbf{A}} = \hat{\mathbf{\Phi}}^{\mathrm{T}}\mathbf{A}\hat{\mathbf{\Phi}}$ for any matrix $\mathbf{A}$, where $\hat{\mathbf{\Phi}}$ is the matrix composed of nondimensional eigenmodes $\hat{\boldsymbol{\varphi}} = [-h/b \ \alpha \ \beta]^{\mathrm{T}}$. Also, T_1 to T_{19} are given by

$$T_1 = -\tfrac{1}{3}(2 + x_c^2)\sqrt{1 - x_c^2} + x_c \arccos x_c,$$

$$T_3 = -\frac{1}{8}(1 - x_c^2)(5x_c^2 + 4) + \frac{1}{4}x_c(7 + 2x_c^2)\sqrt{1 - x_c^2}\arccos x_c - \left(x_c^2 + \frac{1}{8}\right)(\arccos x_c)^2,$$

$$T_4 = x_c\sqrt{1 - x_c^2} - \arccos x_c,$$

$$T_5 = -(1 - x_c^2) - (\arccos x_c)^2 + 2x_c\sqrt{1 - x_c^2}\arccos x_c,$$

$$T_7 = \frac{1}{8}x_c(7 + 2x_c^2)\sqrt{1 - x_c^2} - \left(x_c^2 + \frac{1}{8}\right)\arccos x_c,$$

$$T_8 = \tfrac{1}{3}(1 + 2x_c^2)\sqrt{1 - x_c^2} + x_c \arccos x_c,$$

$$T_9 = \frac{1}{2}\left[\frac{1}{3}(1-x_c^2)^{3/2} + x_a T_4\right],$$

$$T_{10} = \sqrt{1-x_c^2} + \arccos x_c,$$

$$T_{11} = (2-x_c)\sqrt{1-x_c^2} + (1-2x_c)\arccos x_c,$$

$$T_{12} = \sqrt{1-x_c^2}(2+x_c) - (2x_c+1)\arccos x_c,$$

$$T_{13} = -\tfrac{1}{2}[T_7 + (x_c - x_a)T_1],$$

$$T_{15} = T_4 + T_{10},$$

$$T_{16} = T_1 - T_8 - (x_c - x_a)T_4 + \tfrac{1}{2}T_{11},$$

$$T_{17} = -2T_9 - T_1 + \left(x_a - \frac{1}{2}\right)T_4,$$

$$T_{18} = T_5 - T_4 T_{10},$$

$$T_{19} = -\tfrac{1}{2}T_4 T_{11},$$

Appendix C. Aerodynamic matrix

$$\mathbf{A}(s,U) = \rho U^2 b^2(\hat{\mathbf{M}}_{nc}p^2 + \hat{\mathbf{B}}_{nc}p + \hat{\mathbf{K}}_{nc} + \hat{\mathbf{R}}_{s1}C(p) + \hat{\mathbf{R}}_{s2}pC(p)),$$

where $p = sb/U$, and

$$C(p) = \frac{\mathrm{K}_1(p)}{\mathrm{K}_0(p) + \mathrm{K}_1(p)}$$

is the generalized Theodorsen function, with $\mathrm{K}_0(p)$ and $\mathrm{K}_1(p)$ being the modified Bessel functions.

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